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COINCIDENCE AND COMMON FIXED POINT THEOREMS IN Ƒ-METRIC SPACES

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ABSTRACT

Recently, the concept of Ƒ metric space has been introduced and have been defined a natural topology in this spaces by Jleli and Samet[1]. Furthermore, a new style of Banach contraction principle has been given in the Ƒ metric spaces. In this paper, we prove some coincidence and common fixed point theorems in Ƒ metric spaces.

Keywords: Ƒ Metric Space, Contraction Mapping, Coincidence Point, Common Fixed Point Theorem

PRELIMINARIES, BACKGROUND AND NOTATION

Lately, some authors have given various generalizations of metric spaces. This situation allows authors to find new work areas. Czerwik [2] described the concept of b–metric. Khamsi and Hussain [3] reintroduced the b–metric concept and they called it metric-type. Fagin et al. [4] presented the concept of s–relaxed$_p$ metric. Here, b–metric concept is more general than s–relaxed$_p$ metric concept [5]. G..ahler [6] began the concept of a 2–metric. This metric function defined on the product set $X \times X \times X$. The notion of 2–metric is a generalization of the concept of usual metric. Mustafa and Sims [7] introduced the notion of $\mathcal{G}$–metric space. The notion is more general than the usual metric space. Matthews [8] defined the concept of a partial metric. Jleli and Samet [9] introduced the notion of JS–metric. Third condition of this metric gaves by lim sup – condition. Currently, Jleli and Samet [10] have introduced the Ƒ-metric space concept. After that Alnaser et al. [11] defined relation theoretic contraction and proved some generalized fixed point theorems in this metric spaces. In this work, we give some fixed point theorems in Ƒ-metric spaces, some examples provide the theorems. Also, we prove some common fixed point theorems in the spaces.

Definition 1 Let Ƒ be the set of functions $g: (0, \infty) \to \mathbb{R}$. This function provides the following conditions.

$Ƒ_1$: g is non-decreasing function

$Ƒ_2$: $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} g(a_n) = -\infty$, for every sequence $\{a_n\} \subseteq (0 +\infty)$ [1].

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Definition 2 Let $X \neq \emptyset$, $D: X \times X \to [0,\infty)$ be a given mapping and there exists $g \in Ƒ$ and $\gamma \in [0,\infty)$. If the following conditions are satisfied, then D is called as an Ƒ -metric on X. The pair (X, D) is called as an Ƒ -metric space.

D_1 : $(a,b) \in X \times X, D(a,b) = 0 \Leftrightarrow a = b$,

D_2 : $D(a,b) = D(b,a)$ for all $a, b \in X$,

D_3 : For every $a, b \in X$, for every $n \in \mathbb{N}$, $n \geq 2$ and for every $(t_i)_{i=1}^{n} \subset X$ with

$(t_1, t_{n_0}) = (a,b)$ we have $D(a,b) > 0 \Rightarrow g(D(a,b)) \leq g(\Sigma_{i=1}^{n-1} D((t_i, t_{i+1})) + \gamma$ [1].

Remark 3 Any metric space is an Ƒ-metric space. The converse of this proposition is false [1].

Example 4 Let $\mathbb{R}^+$ be the set of positive real numbers. $D: \mathbb{R}^+ \times \mathbb{R}^+ \to [0,\infty)$ be the mapping and for all $a, b \in \mathbb{R}^+$,

$$D(a,b) = \begin{cases} (|a-b|)^2, & a,b \in [0,3] \\ |a-b|, & a,b \notin [0,3] \end{cases}.$$

Then D is an Ƒ-metric with $g(a) = \ln a$ and $\gamma = ln3$ [1].

Example 5 Let $\mathbb{N}$ be the set of natural numbers, $D: \mathbb{N} \times \mathbb{N} \to [0,\infty)$ be the mapping and for all $a, b \in \mathbb{N}$,

$$D(a,b) = \begin{cases} exp(|a-b|), & a \neq b \\ 0, & a = b \end{cases}.$$

Then D is an Ƒ-metric with $g(a) = -\frac{1}{a}$, $a > 0$ and $\gamma = 1$ [1].

Definition 6 Suppose that D be an Ƒ-metric on X. Let $\{a_n\}$ be a sequence in X.

i. If $\{a_n\}$ is convergent to element according to the Ƒ-metric D, then the sequence $\{a_n\}$ is Ƒ –convergent to element a.
ii. If $\lim\limits_{m,n\to\infty} D(a_n, a_m) = 0$ then the sequence $\{a_n\}$ is Ƒ-Cauchy.
iii. If every Ƒ-Cauchy sequence in X is convergent, then (X, D) is Ƒ-complete [1].

Definition 7 Let T and S be self maps of a set X. If $y = Tx = Sx$ for some $x \in X$ then y is said to be a point of coincidence and x is said to be a coincidence point of T and S.

If $x = Tx = Sx$ for some $x \in X$ then x is said to be a common fixed point of T and S [10].

Remark 8 If T and S are weakly compatible, that is, they are commuting at their coincidence point on X, then the point of coincidence y is the unique common fixed point of these maps [10].

Theorem 9 Let $X \neq \emptyset$, D be an Ƒ-metric on X and $f: X \to X$ be a given mapping. Assume that the Ƒ-metric space (X, D) is Ƒ-complete and there exists $\alpha \in (0,1)$ such that $D(f(a), f(b)) \leq$

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$\alpha D(a,b)$ for $a,b \in X$. Then f has a unique fixed point $a^* \in X$. Furthermore, the sequence $\{a_n\} \subset X$ defined by $a_{n+1} = f(a_n)$, $n \in \mathbb{N}$ is Ƒ-convergent to a^*, for any $a_0 \in X$ [1].

MAIN RESULTS

In this section, we give generalizations of some known fixed point theorems in the setting of the Ƒ-metric spaces.

Theorem 10 Let $X \neq \emptyset$, D be an Ƒ-metric on X. Assume the two mappings S, T: $X \to X$ satisfies the following conditions

i. For all $a,b \in X$, $D(T(a),T(b)) \leq k.u(a,b)$ where $0 \leq k < 1$ is a constant and $u(a,b) \in \{D(Sa,Sb), D(Sa,Ta), D(Sb,Tb), \frac{1}{2}[D(Sa,Tb) + D(Sb,Ta)]\}$,

ii. $T(X) \subset$ S(X),

iii. $T(X)$ or S(X) be Ƒ-complete subspace of X.

Then T and S have a unique point of coincidence in X.

Moreover, if T and S are weakly compatible, then they have a unique common fixed point in X.

Proof Let $g \in$ Ƒ, $\gamma \in [0,\infty)$ be such that for every $a,b \in X$ for every $n \in \mathbb{N}$, $n \geq 2$ and for every $(t_i)_{i=1}^{n} \subset X$ with $(t_1, t_{n_0}) = (a,b)$, we have

$$D(a,b) > 0 \Rightarrow g(D(a,b)) \leq g(\Sigma_{i=1}^{n-1} D((t_i, t_{i+1})) + \gamma.$$

From Ƒ$_2$, for every sequence $a_n \subseteq (0,+\infty)$, there exists a $\delta > 0$ such that $n \to \infty$ $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} g(a_n) = -\infty$ and $0 < a < \delta \Rightarrow g(a) < g(\delta) - \gamma$.

Let a_0, $a_1 \in X$ be arbitrary and $\{a_n\} \subset X$ be the sequence defined by Sa_{n+1}=$Ta_n = b_n$ for $n \in \mathbb{N}$. We have that

$D(b_n, b_{n+1})$= $D(Ta_n, Ta_{n+1}) \leq k.u(a_n, a_{n+1})$, for all $n \in \mathbb{N}$. …*

Now, we have to consider the following cases,

If $u(a_n, a_{n+1}) = D(b_{n-1}, b_n)$ then clearly (*) holds.

If $u(a_n, a_{n+1}) = D(b_n, b_{n+1})$ then $D(b_n, b_{n+1}) = 0$ and (*) is immediate.

Finally, suppose that $u(a_n, a_{n+1}) = \frac{1}{2} D(b_{n-1}, b_{n+1})$.

Then,

$$D(b_n, b_{n+1}) \leq \frac{k}{2}.D(b_{n-1}, b_{n+1}) \leq \frac{k}{2}.D(b_{n-1}, b_n) + \frac{1}{2}.D(b_n, b_{n+1})$$

Holds, and we prove (*).

We have

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$$D(b_n, b_{n+1}) \leq k^n D(b_0, b_1).$$

Thus for all n and p,

$$D(b_n, b_{n+p}) \leq D(b_n, b_{n+1}) + D(b_{n+1}, b_{n+2}) + \cdots + D(b_{n+p-1}, b_{n+p})$$

$$\leq k^n + k^{n+1} + \cdots + k^{n+p-1})D(b_0, b_1)$$

$$\leq \frac{k^n}{1-k} D(b_0, b_1)$$

holds.

Since $\lim_{n\to\infty} \frac{k^n}{1-k} D(b_0, b_1) = 0$, there exists a $n_0 \in \mathbb{N}$ such that $0 < \frac{k^n}{1-k} D(b_0, b_1) < \delta, \quad n \geq n_0$. From conditions $0 < b < \delta \Rightarrow g(b) < g(\delta) - \gamma$ and g is non-decreasing.

$$g(\Sigma_{i=n}^{n+p-1} D(b_i, b_{i+1})) \leq g(\frac{k^n}{1-k} D(b_0, b_1)) < g(\delta) - \gamma, n \geq n_0 \text{ ....**}$$

Using conditions (D_3) and (**),

$$D(b_n, b_{n+p}) > 0, n \geq n_0 \Rightarrow g(D(b_n, b_{n+p})) \leq g(\Sigma_{i=n}^{n+p-1} D(b_i, b_{i+1})) + \gamma < g(\delta).$$

Thus we obtain that $D(b_n, b_{n+p}) < \delta, n \geq n_0$ by $(\mathcal{F}_1)$. It is seen that this sequence $\{b_n\}$ is an $\mathcal{F}$ −Cauchy.

Since the range of S contains the range of T and the range of at least one is $\mathcal{F}$-complete, there exists a $c \in S(X)$ such that $\lim_{n\to\infty} D(Sa_n, c) = 0$. Hence there exists a sequence (x_n) in $[0, +\infty)$ such that $x_n \to 0$ and $D(Sa_n, c) \leq x_n$. On the other hand, we can find $d \in X$ such that $Sd = c$. Let us show that $Td = c$. We suppose that $D(Td, c) > 0$.

From the condition (D_3),

$$g(D(Td, c)) \leq g(D(Td, Tb_n)) + D(Tb_n, c) + \gamma, n \in \mathbb{N}.$$

Using condition of theorem and g is non-decreasing,

$$g(D(Td, c)) \leq g[k(D(Td, c) + 2D(Tb_n, b_n) + D(b_n, c))] + \gamma, n \in \mathbb{N}.$$

Otherwise, using $\lim_{n\to\infty} b_n = 0 \Leftrightarrow \lim_{n\to\infty} g(b_n) = -\infty$ and $\lim_{n\to\infty} [D(Td, c) + 2D(Tb_n, b_n) + D(b_n, c)] = 0$, we obtain that

$$\lim_{n\to\infty} g(k[D(Td, c) + 2D(Tb_n, b_n) + D(b_n, c)]) + \gamma = -\infty.$$

This is a contradiction. Consequently, $D(Td, c) = 0$ i.e. Td$= c$ and so c is a point of coincidence of T and S.

If c_1 is another point of coincidence then there is $d_1 \in X$ with $\mathrm{T}d_1 = \mathrm{S}d_1 = c_1$. Therefore $D(c, c_1) = D(Td, \mathrm{T}d_1) \leq ku(d, d_1)$, where

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$$u(d, d_1) \in \{D(Sd, Sd_1), D(Sd, \mathrm{T}d), D(Sd_1, \mathrm{T}d_1), \frac{1}{2}[D(Sd, \mathrm{T}d_1) + D(Sd_1, Td)]\}$$
$$= \{0, D(c, c_1)\}.$$

Hence $D(c, c_1) = 0$ that is $c = c_1$.

If T and S are weakly compatible, then it is obvious that z is unique common fixed point of T and S.

Theorem 11 Let $X \neq \emptyset$, D be an Ƒ-metric on X. Assume the two mappings S, T: $X \to X$ satisfies the following conditions

i. For all $a, b \in X$, $D(T(a), T(b)) \leq k.u(a, b)$ where $0 \leq k < 1$ is a constant and $u(a, b) \in \{D(Sa, Sb), \frac{1}{2}[D(Sa, Ta) + D(Sb, Tb)], \frac{1}{2}[D(Sa, Tb) + D(Sb, Ta)]\}$,
ii. $T(X) \subset$ S(X),
iii. $T(X)$ or S(X) be Ƒ-complete subspace of X.

Then T and S have a unique point of coincidence in X.

Moreover, if T and S are weakly compatible, then they have a unique common fixed point in X.

Proof Let $g \in$ Ƒ, $\gamma \in [0, \infty)$ be such that for every $a, b \in X$ for every $n \in \mathbb{N}$, $n \geq 2$ and for every $(t_i)_{i=1}^{n} \subset X$ with $(t_1, t_{n_0}) = (a, b)$, we have

$$D(a, b) > 0 \Rightarrow g(D(a, b)) \leq g(\Sigma_{i=1}^{n-1} D((t_i, t_{i+1})) + \gamma.$$

From Ƒ$_2$, for every sequence $a_n \subseteq (0, +\infty)$, there exists a $\delta > 0$ such that $n \to \infty$ $\lim_{n\to\infty} a_n = 0 \Leftrightarrow \lim_{n\to\infty} g(a_n) = -\infty$ and $0 < a < \delta \Rightarrow g(a) < g(\delta) - \gamma$.

Let $a_0, a_1 \in X$ be arbitrary and $\{a_n\} \subset X$ be the sequence defined by Sa_{n+1}=$Ta_n = b_n$ for $n \in \mathbb{N}$. We have that

$D(b_n, b_{n+1})$= $D(Ta_n, Ta_{n+1}) \leq k.u(a_n, a_{n+1})$, for all $n \in \mathbb{N}$. …*

We first show that $D(b_n, b_{n+1}) \leq k.D(b_{n-1}, b_n)$, for all $n \in \mathbb{N}$. Notice that

$D(b_n, b_{n+1}) = D(Ta_n, Ta_{n+1}) \leq k.u(a_n, a_{n+1})$, for all $n \in \mathbb{N}$.

As in the above theorem, we have to consider the following cases,

If $u(a_n, a_{n+1}) = D(b_{n-1}, b_n)$, $u(a_n, a_{n+1}) = \frac{1}{2}[D(b_{n-1}, b_n) + D(b_n, b_{n+1})]$ and $u(a_n, a_{n+1}) = \frac{1}{2} D(b_{n-1}, b_{n+1})$.

First and third have been shown in the proof of Theorem 2.1. Consider only the second case.

If $u(a_n, a_{n+1}) = \frac{1}{2}[D(b_{n-1}, b_n) + D(b_n, b_{n+1})] \leq \frac{k}{2} D(b_{n-1}, b_n) + \frac{1}{2} D(b_n, b_{n+1})$.

Hence (***) holds.

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As in the proof of above theorem, we show that (b_n) is an $\mathcal{F}-$ Cauchy sequence. Then there exist $c \in S(X)$, $d \in X$ and (x_n) in $[0, +\infty)$ such that $Sd = c, D(Sa_n, c) \leq x_n, x_n \to 0$. We have $D(Td, c) \leq D(Td, Ta_n) + D(Ta_n, c) \leq u(a_n, d) + x_{n+1}$ for all n.

Since $u(a_n, d) \in \{D(Sa_n, Sd), \frac{1}{2}[D(Sa_n, Ta_n) + D(Sd, Td)], \frac{1}{2}[D(Sa_n, Td) + D(Sd, Ta_n)]\}$, at least one of three cases holds for all n. Consider only the case of $u(a_n, d) = \frac{1}{2}[D(Sa_n, Ta_n) + D(Sd, Td)]$ because the other two cases have shown that the proof of Theorem 2.1. It is satisfied that

$$(Td, c) \leq \frac{1}{2}[D(Sa_n, Ta_n) + D(Sd, Td)] + x_{n+1}$$

$$\leq \frac{1}{2}D(Sa_n, c) + \frac{1}{2}D(Ta_n, c) + \frac{1}{2}D(Td, c) + x_{n+1}$$

$$\leq \frac{1}{2}x_n + \frac{1}{2}D(Td, c) + \frac{3}{2}x_{n+1}$$

$$\leq \frac{1}{2}D(Td, c) + 2x_n,$$

that is, $(Td, c) \leq 4x_n$. Since $4x_n \to 0$, then $Td = c$. Hence, c is a point of coincidence of T and S. The uniqueness of c as in the above. Also, if S and T are weakly compatible, then it is obvious that c is unique common fixed point of T and S.

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